Assignment 1 - Solutions

Hanna Ek

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Problem 1

A

1

$$\frac{1}{\sqrt{2}}(X+Z) = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} := H$$

2

$$\begin{aligned} HXH &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \\ &= \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} := Z \end{aligned}$$

$$\begin{aligned} HYH &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}-i & i\\ i & i\end{bmatrix} = \\ &= \frac{1}{2}\begin{bmatrix}0 & 2i\\ -2i & 0\end{bmatrix} = \begin{bmatrix}0 & i\\ -i & 0\end{bmatrix} = -Y. \end{aligned}$$

$$\begin{aligned} HZH &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} = \\ &= \frac{1}{2}\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = X. \end{aligned}$$

3

$$\left.\begin{aligned} XZ &= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \\ -ZX &= -\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \end{aligned}\right\} \implies XZ = -ZX$$

$$\left.\begin{aligned} XY &= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = \begin{bmatrix}i & 0\\0 & -i\end{bmatrix} \\ -YX &= -\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = -\begin{bmatrix}-i & 0\\0 & i\end{bmatrix}\end{aligned}\right\} \implies XY = -YX$$

$$\left.\begin{aligned} ZY &= \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = \begin{bmatrix}0 & -i\\-i & 0\end{bmatrix} \\ -YZ &= -\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = -\begin{bmatrix}0 & i\\i & 0\end{bmatrix}\end{aligned}\right\} \implies ZY = -YZ$$

4

$$\left.\begin{aligned} TZ &= \begin{bmatrix}1 & 0\\0 & e^{i\pi/4}\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -e^{i\pi/4}\end{bmatrix} \\ ZT &= \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & e^{i\pi/4}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -e^{i\pi/4}\end{bmatrix}\end{aligned}\right\} \implies TZ = ZT$$

5

$(X \otimes X)CNOT = CNOT(X \otimes I)$

To prove this let's look at how this acts on the computational basis.

$$\begin{aligned}
|00\rangle &\xrightarrow{(X\otimes I)} |10\rangle \xrightarrow{CNOT} |11\rangle \\
|01\rangle &\xrightarrow{(X\otimes I)} |11\rangle \xrightarrow{CNOT} |10\rangle \\
|10\rangle &\xrightarrow{(X\otimes I)} |00\rangle \xrightarrow{CNOT} |00\rangle \\
|11\rangle &\xrightarrow{(X\otimes I)} |01\rangle \xrightarrow{CNOT} |01\rangle
\end{aligned}$$

$$\begin{aligned}
|00\rangle &\xrightarrow{CNOT} |00\rangle \xrightarrow{(X\otimes X)} |11\rangle \\
|01\rangle &\xrightarrow{CNOT} |01\rangle \xrightarrow{X\otimes X)} |10\rangle \\
|10\rangle &\xrightarrow{CNOT} |11\rangle \xrightarrow{X\otimes X)} |00\rangle \\
|11\rangle &\xrightarrow{CNOT} |10\rangle \xrightarrow{X\otimes X)} |01\rangle
\end{aligned}$$

Which shows that the two circuits are equivalent.

$(I \otimes X)CNOT = CNOT(I \otimes X)$

We again look at how this acts on the computational basis.

$$\begin{aligned}
|00\rangle &\xrightarrow{(I\otimes X)} |01\rangle \xrightarrow{CNOT} |01\rangle \\
|01\rangle &\xrightarrow{(I\otimes X)} |00\rangle \xrightarrow{CNOT} |00\rangle \\
|10\rangle &\xrightarrow{(I\otimes X)} |11\rangle \xrightarrow{CNOT} |10\rangle \\
|11\rangle &\xrightarrow{(I\otimes X)} |10\rangle \xrightarrow{CNOT} |11\rangle
\end{aligned}$$

$$\begin{aligned}
|00\rangle &\xrightarrow{CNOT} |00\rangle \xrightarrow{(I\otimes X)} |01\rangle \\
|01\rangle &\xrightarrow{CNOT} |01\rangle \xrightarrow{(I\otimes X)} |00\rangle \\
|10\rangle &\xrightarrow{CNOT} |11\rangle \xrightarrow{(I\otimes X)} |10\rangle \\
|11\rangle &\xrightarrow{CNOT} |10\rangle \xrightarrow{(I\otimes X)} |11\rangle
\end{aligned}$$

Which shows that the two circuits are equivalent.

B

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The computational basis for a two-qubit system in vector representation is,

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

From this it follows that

$$CZ\,|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle\,.$$

Repeating the same computations on the other computational basis states gives,

$$CZ\,|00\rangle = |00\rangle \quad CZ\,|01\rangle = |01\rangle \quad CZ\,|10\rangle = |10\rangle \quad CZ\,|11\rangle = -\,|11\rangle\,.$$

1

To show $CZ_{1,2} = CZ_{2,1}$ we show that both act in the same way on the computational basis.

$$
\begin{aligned}
|00\rangle \xrightarrow{CZ_{1,2}} |00\rangle \xleftarrow{CZ_{2,1}} |00\rangle \\
|01\rangle \xrightarrow{CZ_{1,2}} |01\rangle \xleftarrow{CZ_{2,1}(Z|0\rangle=|0\rangle)} |01\rangle \\
|10\rangle \xrightarrow{CZ_{1,2}(Z|0\rangle=|0\rangle)} |10\rangle \xleftarrow{CZ_{2,1}} |10\rangle \\
|11\rangle \xrightarrow{CZ_{1,2}(Z|1\rangle=-|1\rangle)} -|11\rangle \xleftarrow{CZ_{2,1}(Z|1\rangle=-|1\rangle)} |11\rangle
\end{aligned}
$$

2

Let $|\psi\rangle = |01\rangle$.

$$CNOT_{1,2}\,|\psi\rangle = |01\rangle \neq CNOT_{2,1}\,|\psi\rangle = |11\rangle \implies CNOT_{1,2} \neq CNOT_{2,1}.$$

3

To show $(I \otimes H)CZ_{1,2}(I \otimes H) = CNOT_{1,2}$ we show that both acts the same on the computational basis.

First consider $(I \otimes H)CZ_{1,2}(I \otimes H)$:

- $|\mathbf{00}\rangle \xrightarrow{(I\otimes H)} |0\rangle|+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) \xrightarrow{CZ_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) \xrightarrow{(I\otimes H)}$
  $\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |0\rangle|-\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |00\rangle - |01\rangle) = \frac{1}{2}2\,|00\rangle = |\mathbf{00}\rangle$
- $|\mathbf{01}\rangle \xrightarrow{(I\otimes H)} |0\rangle|-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle) \xrightarrow{CZ_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle) \xrightarrow{(I\otimes H)}$
  $\frac{1}{\sqrt{2}}(|0\rangle|+\rangle - |0\rangle|-\rangle) = ... = |\mathbf{01}\rangle$
- $|\mathbf{10}\rangle \xrightarrow{(I\otimes H)} |1\rangle|+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle) \xrightarrow{CZ_{1,2}} \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) \xrightarrow{(I\otimes H)}$
  $\frac{1}{\sqrt{2}}(|1\rangle|+\rangle - |1\rangle|-\rangle) = ... = |\mathbf{11}\rangle$
- $|\mathbf{11}\rangle \xrightarrow{(I\otimes H)} |1\rangle|-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) \xrightarrow{CZ_{1,2}} \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle) \xrightarrow{(I\otimes H)}$
  $\frac{1}{\sqrt{2}}(|1\rangle|+\rangle + |1\rangle|-\rangle) = ... = |\mathbf{10}\rangle$

We know from before that,

$CNOT_{1,2}\,|00\rangle \mapsto |00\rangle\,, CNOT_{1,2}\,|01\rangle \mapsto |01\rangle\,, CNOT_{1,2}\,|10\rangle \mapsto |11\rangle$ and $CNOT_{1,2}\,|11\rangle \mapsto |10\rangle\,.$

combining the two results gives that $(I \otimes H)CZ_{1,2}(I \otimes H) = CNOT_{1,2}$.

4

To show $(H \otimes H)CNOT_{1,2}(H \otimes H) = CNOT_{2,1}$ we show that both acts the same on the computational basis.

It can easily be seen that:

$CNOT_{2,1}|00\rangle \mapsto |00\rangle, CNOT_{2,1}|01\rangle \mapsto |11\rangle, CNOT_{2,1}|10\rangle \mapsto |10\rangle$ and $CNOT_{2,1}|11\rangle \mapsto |01\rangle$.

Next show that $(H \otimes H)CNOT_{1,2}(H \otimes H)$ acts the same on the computational basis.

- $|\mathbf{00}\rangle \xrightarrow{(H\otimes H)} |+\rangle|+\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \xrightarrow{CNOT_{1,2}} \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) \xrightarrow{(H\otimes H)} \frac{1}{2}(|+\rangle|+\rangle + |+\rangle|-\rangle + |-\rangle|-\rangle + |-\rangle|+\rangle) = ... = |\mathbf{00}\rangle$
- $|\mathbf{01}\rangle \xrightarrow{(H\otimes H)} |+\rangle|-\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \xrightarrow{CNOT_{1,2}} \frac{1}{2}(|00\rangle - |01\rangle + |11\rangle - |10\rangle) \xrightarrow{(H\otimes H)} \frac{1}{2}(|+\rangle|+\rangle - |+\rangle|-\rangle + |-\rangle|-\rangle - |-\rangle|+\rangle) = ... = |\mathbf{11}\rangle$
- $|\mathbf{10}\rangle \xrightarrow{(H\otimes H)} |-\rangle|+\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \xrightarrow{CNOT_{1,2}} \frac{1}{2}(|00\rangle + |01\rangle - |11\rangle - |10\rangle) \xrightarrow{(H\otimes H)} \frac{1}{2}(|+\rangle|+\rangle + |+\rangle|-\rangle - |-\rangle|-\rangle - |-\rangle|+\rangle) = ... = |\mathbf{10}\rangle$
- $|\mathbf{11}\rangle \xrightarrow{(H\otimes H)} |-\rangle|-\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \xrightarrow{CNOT_{1,2}} \frac{1}{2}(|00\rangle - |01\rangle - |11\rangle + |10\rangle) \xrightarrow{(H\otimes H)} \frac{1}{2}(|+\rangle|+\rangle - |+\rangle|-\rangle - |-\rangle|-\rangle + |-\rangle|+\rangle) = ... = |\mathbf{01}\rangle$

Problem 2

A

Express $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in the $\pm$-basis. Expressing this in the computational basis,

$$\begin{aligned}
|{+}{+}\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
|{+}{-}\rangle &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\
|{-}{+}\rangle &= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \\
|{-}{-}\rangle &= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)
\end{aligned}$$

From this we can see that $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|{+}{+}\rangle - |{-}{-}\rangle)$.

Express $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ in the $(|{+}{+}{+}\rangle, |{+}{+}{-}\rangle, ..., |{-}{-}{-}\rangle)$-basis. With similar a computation as above we get that $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \frac{1}{2}(|{+}{+}{+}\rangle + |{+}{-}{-}\rangle + |{-}{+}{-}\rangle + |{-}{-}{+}\rangle)$.

B

$$\begin{aligned}
CNOT\,|{+}{+}\rangle = CNOT\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \\
= \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) = |{+}{+}\rangle = |+\rangle \otimes |+\rangle\,.
\end{aligned}$$

Yes, it is a product state, since the state is equal to $|+\rangle \otimes |+\rangle$.

C

$$CZ\,|{+}{+}\rangle = CNOT\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle).$$

This state is not a product state. To show this consider two states $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\phi\rangle = c|0\rangle + d|1\rangle$, their joint states is,

$$|\psi\rangle \otimes |\phi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \implies$$

$$ac = 1 = bc \implies a = b \text{ (since } c \neq 0) \text{ }, ad = 1 \overset{a=b}{=} bd = -1 \implies 1 = -1.$$

Which is a contradiction.

D

Let $|\psi\rangle = a|0\rangle + b|1\rangle$, compute $CNOT|\psi\rangle|-\rangle$,

$$\begin{aligned} CNOT|\psi\rangle \otimes |-\rangle &= CNOT\Big(\frac{1}{\sqrt{2}}(a|00\rangle - a|01\rangle + b|10\rangle - b|11\rangle)\Big) \\ &= \frac{1}{\sqrt{2}}(a|00\rangle - a|01\rangle + b|11\rangle - b|10\rangle). \end{aligned}$$

The state is a product state, let $|\phi\rangle = a|0\rangle - b|1\rangle$ and $|\gamma\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ then it follow that $|\phi\rangle \otimes |\gamma\rangle = \frac{1}{\sqrt{2}}(a|00\rangle - a|01\rangle + b|11\rangle - b|10\rangle)$.

E

Show that $SWAP = CNOT_{1,2}CNOT_{2,1}CNOT_{1,2}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle$, where $|\psi\rangle, |\phi\rangle$ are single qubit states. To show this it is sufficient to show it for the computational basis.

$$\begin{aligned} |0\rangle \otimes |0\rangle = |00\rangle &\xrightarrow{CNOT_{1,2}} |00\rangle \xrightarrow{CNOT_{2,1}} |00\rangle \xrightarrow{CNOT_{1,2}} |00\rangle = |0\rangle \otimes |0\rangle \\ |0\rangle \otimes |1\rangle = |01\rangle &\xrightarrow{CNOT_{1,2}} |01\rangle \xrightarrow{CNOT_{2,1}} |11\rangle \xrightarrow{CNOT_{1,2}} |10\rangle = |1\rangle \otimes |0\rangle \\ |1\rangle \otimes |0\rangle = |10\rangle &\xrightarrow{CNOT_{1,2}} |11\rangle \xrightarrow{CNOT_{2,1}} |01\rangle \xrightarrow{CNOT_{1,2}} |01\rangle = |0\rangle \otimes |1\rangle \\ |1\rangle \otimes |1\rangle = |11\rangle &\xrightarrow{CNOT_{1,2}} |10\rangle \xrightarrow{CNOT_{2,1}} |10\rangle \xrightarrow{CNOT_{1,2}} |11\rangle = |1\rangle \otimes |1\rangle \end{aligned}$$

F

The unitary matrix describing the transformation:

$$|00\rangle \mapsto |00\rangle, |01\rangle \mapsto -|01\rangle, |10\rangle \mapsto -|10\rangle, |11\rangle \mapsto |11\rangle.$$

is

$$\mathcal{U}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = Z \otimes Z.$$

The unitary matrix describing the transformation:

$$|00\rangle \mapsto |00\rangle, |01\rangle \mapsto |01\rangle, |10\rangle \mapsto |11\rangle\, |11\rangle \mapsto -|10\rangle.$$

is

$$\mathcal{U}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = CNOT \cdot CZ = CNOT(I \otimes H)CNOT(I \otimes H).$$

Problem 3

A

Show that $H^{\otimes 2}|00\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.

$$\begin{aligned} H^{\otimes 2}|00\rangle &= (H \otimes H)(|0\rangle \otimes |0\rangle) = H|0\rangle \otimes H|0\rangle = \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle). \end{aligned}$$

This can be rewritten as $H^{\otimes 2}|00\rangle = \frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle) = \frac{1}{\sqrt{2^2}}\sum_{x\in\{0,1\}^2}|x\rangle$. We want to use this to prove that in general it holds that $H^{\otimes n}|0^n\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle$.

For any $n > 2$ we have,

$$\begin{aligned} H^{\otimes n}|0^n\rangle &= (H \otimes H \otimes \cdots H \otimes H)(|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle) = H|0\rangle \otimes H|0\rangle \otimes \cdots \otimes H|0\rangle = \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes H^{\otimes n-1}|0^{n-1}\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2^{n-1}}}\sum_{x\in\{0,1\}^{n-1}}|x\rangle = \\ &= \frac{1}{\sqrt{2^n}}|0\rangle \otimes \sum_{x\in\{0,1\}^{n-1}}|x\rangle + |1\rangle \otimes \sum_{x\in\{0,1\}^{n-1}}|x\rangle = \\ &= \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle\,. \end{aligned}$$

B

For $b \in \{0,1\}$ the Hadamard on the computational basis can be written as,

$$H|b\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^b|1\rangle) = \frac{1}{\sqrt{2}}\sum_{y\in\{0,1\}}(-1)^{b\cdot y}|y\rangle\,.$$

Use this to show $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{x\cdot y}|y\rangle$.

First consider the case where $n = 2$, $x \in \{0,1\}^2 = x_1 \otimes x_2$ where $x_i \in \{0,1\}$ for $i = 1, 2$.

$$\begin{aligned} H^{\otimes 2} |x\rangle &= (H \otimes H)(|x_1\rangle \otimes |x_2\rangle) = H|x_1\rangle \otimes H|x_2\rangle = \\ &= \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x_1 \cdot y} |y\rangle \otimes \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x_2 \cdot y} |y\rangle = \\ &= \frac{1}{\sqrt{2^2}} \sum_{y \in \{0,1\}^2} (-1)^{x_1 \cdot y} (-1)^{x_2 \cdot y} |y\rangle = \frac{1}{\sqrt{2^2}} \sum_{y \in \{0,1\}^2} (-1)^{x \cdot y} |y\rangle . \end{aligned}$$

Now let's consider the case where $n > 2$, $x \in \{0,1\}^n = \bigotimes_{i=1}^n x_i$ where $x_i \in \{0,1\}$ for $i = [1, n]$.

$$\begin{aligned} H^{\otimes n} |x\rangle &= (H \otimes H \otimes \cdots \otimes H)(|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle) = H|x_1\rangle \otimes H|x_2\rangle \otimes \cdots \otimes H|x_n\rangle = \\ &= \left(\frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x_1 \cdot y} |y\rangle \right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x_2 \cdot y} |y\rangle \right) \otimes \cdots \otimes \left(\frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x_n \cdot y} |y\rangle \right) = \\ &= \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x_n \cdot y} |y\rangle \otimes H^{\otimes n-1} |\tilde{x}\rangle = \\ &= \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x_n \cdot y} |y\rangle \otimes \frac{1}{\sqrt{2^{n-1}}} \sum_{y \in \{0,1\}^{n-1}} (-1)^{\tilde{x} \cdot y} |y\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x_n \cdot y} (-1)^{\tilde{x} \cdot y} |y\rangle = \\ &= \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x_n \cdot y + \tilde{x} \cdot y} |y\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle , \end{aligned}$$

where $\tilde{x} = \bigotimes_{i=1}^{n-1} x_i$ where $x_i \in \{0,1\}$ for $i = [1, n-1]$.

Problem 4

A

- $|00\rangle \xrightarrow{(H\otimes I)} |+\rangle|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{CZ_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{CZ_{2,1}} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
- $|00\rangle \xrightarrow{(H\otimes H)} |+\rangle|+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) \xrightarrow{CNOT_{2,1}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle + |01\rangle + |10\rangle) \xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle + |01\rangle + |11\rangle) \xrightarrow{(H\otimes H)} |00\rangle$.
- $|001\rangle \xrightarrow{H^{\otimes 3}} |+\rangle|+\rangle|-\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle) \xrightarrow{CNOT_{1,3}} \frac{1}{2\sqrt{2}}(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |101\rangle - |100\rangle + |111\rangle - |110\rangle) \xrightarrow{CNOT_{2,3}} \frac{1}{2\sqrt{2}}(|000\rangle - |001\rangle + |011\rangle - |010\rangle + |101\rangle - |100\rangle + |110\rangle - |111\rangle) = |---\rangle$.
- $|00\rangle \xrightarrow{(H\otimes I)} |+\rangle|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{(T\otimes T^\dagger)} \frac{1}{\sqrt{2}}(|00\rangle + e^{i\pi/4 - i\pi/4}|11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{CNOT_{2,1}} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) \xrightarrow{(I\otimes H)} |00\rangle$

B

1

Since T is unitary it follows that $TT^\dagger = I$, since $T^8 = I$ this can be rewritten as $T(T^7) = I \implies T^7 = T^\dagger$.

2

The unitary performing this operator is $SWAP_{1,2}CNOT_{1,3}CNOT_{2,3}$.

3

The unitary performing this operator is TTH.

4

The unitary performing this operator is $(X \otimes X \otimes I \otimes I)(CNOT_{1,2})(CNOT_{1,3})(CNOT_{2,4})(H \otimes I \otimes I \otimes I)$.

Problem 5

A projective measurment is specified by a collection of projectors $\{P_i\}_{i=1}^k$ satisfies $P_i^2 = P_i$, $P_i P_j = 0$ and $\sum_{i=1}^k P_i = I$.

A

Show that $P_0 = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $P_1 = |01\rangle\langle 01| + |10\rangle\langle 10|$ constitutive a valid projective measurement.

Lets consider P_0 and P_1 in their matrix form:

$$P_0 = |00\rangle\langle 00| + |11\rangle\langle 11| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$P1 = |01\rangle\langle 01| + |10\rangle\langle 10| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

From this it follows by matrix multiplication that $P_0^2 = P_0$, $P_1^2 = P_1$ and $P_0 P_1 = 0$. It is also straight forward to see that $P_0 + P_1 = I$.

Alternative way of showing this.

$$\begin{aligned}
P_0^2 &= (|00\rangle\langle 00| + |11\rangle\langle 11|)(|00\rangle\langle 00| + |11\rangle\langle 11|) = \\
&= |00\rangle \overbrace{\langle 00|00\rangle}^{=1} \langle 00| + |00\rangle \overbrace{\langle 00|11\rangle}^{=0} \langle 11| + |11\rangle \overbrace{\langle 11|00\rangle}^{=0} \langle 00| + |11\rangle \overbrace{\langle 11|11\rangle}^{=1} \langle 11| = \\
&= |00\rangle\langle 00| + |11\rangle\langle 11| = P_0. \\
P_1^2 &= (|01\rangle\langle 01| + |10\rangle\langle 10|)(|01\rangle\langle 01| + |10\rangle\langle 10|) = \\
&= |01\rangle \overbrace{\langle 01|01\rangle}^{=1} \langle 01| + |01\rangle \overbrace{\langle 01|10\rangle}^{=0} \langle 10| + |10\rangle \overbrace{\langle 10|01\rangle}^{=0} \langle 01| + |10\rangle \overbrace{\langle 10|10\rangle}^{=1} \langle 10| = \\
&= |01\rangle\langle 01| + |10\rangle\langle 10| = P_1. \\
P_0 P_1 &= (|00\rangle\langle 00| + |11\rangle\langle 11|)(|01\rangle\langle 01| + |10\rangle\langle 10|) = \\
&= |00\rangle \overbrace{\langle 00|01\rangle}^{=0} \langle 01| + |00\rangle \overbrace{\langle 00|10\rangle}^{=0} \langle 10| + |11\rangle \overbrace{\langle 11|01\rangle}^{=0} \langle 01| + |11\rangle \overbrace{\langle 11|10\rangle}^{=0} \langle 10| = 0.
\end{aligned}$$

$$P_0 + P_1 = (|00\rangle\langle 00| + |11\rangle\langle 11|) + (|01\rangle\langle 01| + |10\rangle\langle 10|) = I$$

B

Measuring state $|{++}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ in projective measurement from task A.

First consider P_0. Probability of seeing outcome 0 is given by,

$$\begin{aligned}
\langle{++}| P_0 |{++}\rangle &= \frac{1}{2}\Big(\langle 00| + \langle 01| + \langle 10| + \langle 11|\Big)\Big(|00\rangle\langle 00| + |11\rangle\langle 11|\Big)\frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big) = \\
&= \frac{1}{4}\Big(\langle 00| + \langle 01| + \langle 10| + \langle 11|\Big) \\
&\Big(|00\rangle \overbrace{\langle 00|00\rangle}^{=1} + |00\rangle \overbrace{\langle 00|01\rangle}^{=0} + |00\rangle \overbrace{\langle 00|10\rangle}^{=0} + |00\rangle \overbrace{\langle 00|11\rangle}^{=0} + \\
&+ |11\rangle \overbrace{\langle 11|00\rangle}^{=0} + |11\rangle \overbrace{\langle 11|01\rangle}^{=0} + |11\rangle \overbrace{\langle 11|10\rangle}^{=0} + |11\rangle \overbrace{\langle 11|11\rangle}^{=1}\Big) = \\
&= \frac{1}{4}\Big(\langle 00| + \langle 01| + \langle 10| + \langle 11|\Big)\Big(|00\rangle + |11\rangle\Big) = ... = \frac{1}{4}(\langle 00|00\rangle + \langle 11|11\rangle) = \frac{1}{2}.
\end{aligned}$$

The post-measurement state after seeing 0 in a measurement is,

$$\begin{aligned}
\frac{P_0 |{++}\rangle}{\sqrt{\langle{++}| P_0 |{++}\rangle}} &= \frac{(|00\rangle\langle 00| + |11\rangle\langle 11|)\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)}{\sqrt{\frac{1}{2}}} = \\
&= ... = \frac{(|00\rangle + |11\rangle)}{\sqrt{2}}.
\end{aligned}$$

Doing the same computation for P_1 gives that $\langle{++}| P_1 |{++}\rangle = \frac{1}{2}$ and $\frac{P_1|{++}\rangle}{\sqrt{\langle{++}|P_1|{++}\rangle}} = \frac{(|01\rangle + |10\rangle)}{\sqrt{2}}$.

If we instead consider measuring the state $\Phi^+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, what is the probability then of seeing 0/1.

We start with computing the probability of seeing 0.

$$\langle\Phi^+| P_0 |\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(\langle 00| + \langle 11|\Big)\Big(|00\rangle\langle 00| + |11\rangle\langle 11|\Big)\frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) =$$
$$= \frac{1}{2}\Big(\langle 00| + \langle 11|\Big)\Big(|00\rangle \overbrace{\langle 00||00\rangle}^{=1} + |00\rangle \overbrace{\langle 00||11\rangle}^{=0} + |11\rangle \overbrace{\langle 11||00\rangle}^{=0} + |11\rangle \overbrace{\langle 11||11\rangle}^{=1}\Big) =$$
$$= \frac{1}{2}\Big(\langle 00| + \langle 11|\Big)\Big(|00\rangle + |11\rangle\Big) = ... = \frac{1}{2}(\langle 00||00\rangle + \langle 11||11\rangle) = \frac{1}{2} + \frac{1}{2} = 1.$$

And the post-measurement state is,

$$\frac{P_0|\Phi^+\rangle}{\sqrt{\langle\Phi^+| P_0 |\Phi^+\rangle}} = \frac{(|00\rangle\langle 00| + |11\rangle\langle 11|)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}{\sqrt{1}} = \frac{(|00\rangle\langle 00| + |11\rangle\langle 11|)(|00\rangle + |11\rangle)}{\sqrt{2}} =$$
$$= ... = \frac{|00\rangle \overbrace{\langle 00||00\rangle}^{=1} + |11\rangle \overbrace{\langle 11||11\rangle}^{=1}}{\sqrt{2}} = \frac{(|00\rangle + |11\rangle)}{\sqrt{2}} = |\Phi^+\rangle.$$

Consequently $\langle\Phi^+| P_1 |\Phi^+\rangle = 0$.

C

The projectors for measuring the first qubit in a 3-qubit state are : $P_0 = |0\rangle\langle 0| \otimes I \otimes I$, $P_1 = |1\rangle\langle 1| \otimes I \otimes I$.

- What are the possible post-measurement states and probabilities of them when measuring the first qubit in GHZ state? Remember $GHZ = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

  We start with computing the probability of seeing 0 and 1.

$$\langle GHZ| P_0 |GHZ\rangle = \frac{1}{\sqrt{2}}\Big(\langle 000| + \langle 111|\Big)\Big(|0\rangle\langle 0| \otimes I \otimes I\Big)\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) =$$
$$= \frac{1}{2}\Big(\langle 000| + \langle 111|\Big)\Big(|0\rangle \overbrace{\langle 0||0\rangle}^{=1} \otimes I|0\rangle \otimes I|0\rangle + |0\rangle \overbrace{\langle 0||1\rangle}^{=0} \otimes I|1\rangle \otimes I|1\rangle\Big) =$$
$$= \frac{1}{2}\Big(\langle 000| + \langle 111|\Big)|000\rangle = \frac{1}{2}\Big(\langle 000||000\rangle + \langle 111||000\rangle\Big) = \frac{1}{2}.$$
$$\langle GHZ| P_1 |GHZ\rangle = \frac{1}{\sqrt{2}}\Big(\langle 000| + \langle 111|\Big)\Big(|1\rangle\langle 1| \otimes I \otimes I\Big)\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) = ... = \frac{1}{2}.$$

  And the post-measurement states are ,

$$\frac{P_0|GHZ\rangle}{\sqrt{\langle GHZ| P_0 |GHZ\rangle}} = \frac{(|0\rangle\langle 0| \otimes I \otimes I)\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)}{\frac{1}{\sqrt{2}}} = |000\rangle$$
$$\frac{P_1|GHZ\rangle}{\sqrt{\langle GHZ| P_1 |GHZ\rangle}} = \frac{(|1\rangle\langle 1| \otimes I \otimes I)\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)}{\frac{1}{\sqrt{2}}} = |111\rangle$$

- What are the possible post-measurement states and probabilities of them when measuring the first qubit in $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ state?

  We start with computing the probability of seeing 0 and 1.

$$
\begin{aligned}
\langle W| P_0 |W\rangle &= \frac{1}{\sqrt{3}}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|0\rangle \langle 0| \otimes I \otimes I\Big)\frac{1}{\sqrt{3}}\Big(|001\rangle + |010\rangle + |100\rangle \Big) = \\
&= \frac{1}{3}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|0\rangle \overbrace{\langle 0| |0\rangle}^{=1} \otimes I |0\rangle \otimes I |1\rangle + \\
&+ |0\rangle \overbrace{\langle 0| |0\rangle}^{=1} \otimes I |1\rangle \otimes I |0\rangle + |0\rangle \overbrace{\langle 0| |1\rangle}^{=0} \otimes I |0\rangle \otimes I |0\rangle) = \\
&= \frac{1}{3}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|001\rangle + |010\rangle \Big) = \\
&= \frac{1}{3}\Big(\overbrace{\langle 001| |001\rangle}^{=1} + \overbrace{\langle 010| |001\rangle}^{=0} + \overbrace{\langle 100| |001\rangle}^{=0} + \\
&+ \overbrace{\langle 001| |010\rangle}^{=1} + \overbrace{\langle 010| |010\rangle}^{=0} + \overbrace{\langle 100| |010\rangle}^{=1} \Big) = \frac{2}{3}. \\
\langle W| P_1 |W\rangle &= \frac{1}{\sqrt{3}}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|1\rangle \langle 1| \otimes I \otimes I\Big)\frac{1}{\sqrt{3}}\Big(|001\rangle + |010\rangle + |100\rangle \Big) = \\
&= \frac{1}{3}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|1\rangle \overbrace{\langle 1| |0\rangle}^{=0} \otimes I |0\rangle \otimes I |1\rangle + \\
&+ |1\rangle \overbrace{\langle 1| |0\rangle}^{=0} \otimes I |1\rangle \otimes I |0\rangle + |1\rangle \overbrace{\langle 1| |1\rangle}^{=1} \otimes I |0\rangle \otimes I |0\rangle) = \\
&= \frac{1}{3}\Big(\langle 001| + \langle 010| + \langle 100| \Big) |100\rangle = \\
&= \frac{1}{3}\Big(\overbrace{\langle 001| |100\rangle}^{=0} + \overbrace{\langle 010| |100\rangle}^{=0} + \overbrace{\langle 100| |100\rangle}^{=1} = \frac{1}{3}.
\end{aligned}
$$

  And the post-measurement states are ,

$$
\begin{aligned}
\frac{P_0 |W\rangle}{\sqrt{\langle W| P_0 |W\rangle}} &= \frac{(|0\rangle \langle 0| \otimes I \otimes I)\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)}{\sqrt{\frac{2}{3}}} = \frac{|001\rangle + |010\rangle}{\sqrt{2}} \\
\frac{P_1 |W\rangle}{\sqrt{\langle W| P_1 |W\rangle}} &= \frac{(|1\rangle \langle 1| \otimes I \otimes I)\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)}{\sqrt{\frac{1}{3}}} = |100\rangle
\end{aligned}
$$

D

Remark this circuit is the same as we looked at in Problem 4.*B*.3.

Assuming that $|\psi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$, the 3-qubit state transforms as,

$$
\begin{aligned}
|\psi\rangle \otimes |0\rangle =& (a|000\rangle + b|010\rangle + c|100\rangle + d|110\rangle) \xrightarrow{CNOT_{1,3}} \\
& (a|000\rangle + b|010\rangle + c|101\rangle + d|111\rangle) \xrightarrow{CNOT_{2,3}} \\
& (a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle) = |\tilde{\psi}\rangle .
\end{aligned}
$$

Intrpducing the notation $|\tilde{\psi}\rangle$.

Since we want to measure only the third qubit in the $3 - qubit$ state in the computational basis the corresponding projectors are: $P_0 = I \otimes I \otimes |0\rangle\langle 0|$ and $P_1 = I \otimes I \otimes |1\rangle\langle 1|$.

We start with computing the probability of seeing 0 and 1.

$$
\begin{aligned}
\langle\tilde{\psi}| P_0 |\tilde{\psi}\rangle =& \Big(a\langle 000| + b\langle 011| + c\langle 101| + d\langle 110|\Big) \\
& \Big(I \otimes I \otimes |0\rangle\langle 0|\Big)\Big(a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle\Big) = \\
& = \Big(a\langle 000| + b\langle 011| + c\langle 101| + d\langle 110|\Big) \\
& \Big(a\Big[I|0\rangle \otimes I|0\rangle \otimes |0\rangle \overbrace{\langle 0||0\rangle}^{=1}\Big] + b\Big[I|0\rangle \otimes I|1\rangle \otimes |0\rangle \overbrace{\langle 0||1\rangle}^{=0}\Big] + \\
& + c\Big[I|1\rangle \otimes I|0\rangle \otimes |0\rangle \overbrace{\langle 0||1\rangle}^{=0}\Big] + d\Big[I|1\rangle \otimes I|1\rangle \otimes |0\rangle \overbrace{\langle 0||0\rangle}^{=1}\Big]\Big) = \\
& = \Big(a\langle 000| + b\langle 011| + c\langle 101| + d\langle 110|\Big)\Big(a|000\rangle + d|110\rangle\Big) = \\
& = ... = \Big(a^2\langle 000||000\rangle + d^2\langle 110||110\rangle\Big) = (a^2 + d^2) \\
\langle\tilde{\psi}| P_1 |\tilde{\psi}\rangle =& \Big(a\langle 000| + b\langle 011| + c\langle 101| + d\langle 110|\Big) \\
& \Big(I \otimes I \otimes |1\rangle\langle 1|\Big)\Big(a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle\Big) = \\
& = \Big(a\langle 000| + b\langle 011| + c\langle 101| + d\langle 110|\Big) \\
& \Big(a\Big[I|0\rangle \otimes I|0\rangle \otimes |1\rangle \overbrace{\langle 1||0\rangle}^{=0}\Big] + b\Big[I|0\rangle \otimes I|1\rangle \otimes |1\rangle \overbrace{\langle 1||1\rangle}^{=1}\Big] + \\
& + c\Big[I|1\rangle \otimes I|0\rangle \otimes |1\rangle \overbrace{\langle 1||1\rangle}^{=1}\Big] + d\Big[I|1\rangle \otimes I|1\rangle \otimes |1\rangle \overbrace{\langle 1||0\rangle}^{=0}\Big]\Big) = \\
& = \Big(a\langle 000| + b\langle 011| + c\langle 101| + d\langle 110|\Big)\Big(b|011\rangle + c|101\rangle\Big) = \\
& = ... = \Big(b^2\langle 000||000\rangle + c^2\langle 110||110\rangle\Big) = (b^2 + c^2)
\end{aligned}
$$

$$\frac{P_0 |\tilde{\psi}\rangle}{\sqrt{\langle\tilde{\psi}| P_0 |\tilde{\psi}\rangle}} = \frac{(I \otimes I \otimes |0\rangle \langle 0|)(a |000\rangle + b |011\rangle + c |101\rangle + d |110\rangle)}{\sqrt{a^2 + d^2}} = \frac{a |000\rangle + b |110\rangle}{\sqrt{a^2 + d^2}}$$

$$\frac{P_1 |\tilde{\psi}\rangle}{\sqrt{\langle\tilde{\psi}| P_1 |\tilde{\psi}\rangle}} = \frac{(I \otimes I \otimes |1\rangle \langle 1|)(a |000\rangle + b |011\rangle + c |101\rangle + d |110\rangle)}{\sqrt{b^2 + c^2}} = \frac{b |011\rangle + c |101\rangle}{\sqrt{b^2 + c^2}}$$

E

The projectors for measuring one qubit in the ±-basis are: $P_+ = |+\rangle \langle +|$ and $P_- = |-\rangle \langle -|$.

The projectors for measuring the first qubit of a 3-qubit state the ±-basis are: $P_+ = |+\rangle \langle +| \otimes I \otimes I$ and $P_- = |-\rangle \langle -| \otimes I \otimes I$.

- What are the possible post-measurement states and probabilities of them when measuring the first qubit in GHZ state? Remember $GHZ = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle$.

  We start with computing the probability of seeing 0 and 1.

$$\begin{aligned}
\langle GHZ| P_+ |GHZ\rangle &= \frac{1}{\sqrt{2}}\Big(\langle 000| + \langle 111| \Big)\Big(|+\rangle \langle +| \otimes I \otimes I\Big)\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle \Big) = \\
&= \frac{1}{2}\Big(\langle 000| + \langle 111| \Big)\Big(|+\rangle \overbrace{\langle +| |0\rangle}^{=\frac{1}{\sqrt{2}}} \otimes I |0\rangle \otimes I |0\rangle + |+\rangle \overbrace{\langle +| |1\rangle}^{=\frac{1}{\sqrt{2}}} \otimes I |1\rangle \otimes I |1\rangle) \\
&= \frac{1}{2\sqrt{2}}\Big(\langle 000| + \langle 111| \Big)\Big(|+\rangle |00\rangle + |+\rangle |11\rangle \Big) = \\
&= \frac{1}{2\sqrt{2}}\Big(\overbrace{\langle 000| |+\rangle |00\rangle}^{=\frac{1}{\sqrt{2}}} + \overbrace{\langle 000| |+\rangle |11\rangle}^{=0} + \overbrace{\langle 111| |+\rangle |00\rangle}^{=0} + \overbrace{\langle 111| |+\rangle |11\rangle}^{=\frac{1}{\sqrt{2}}} \Big) = \\
&= \frac{1}{2\sqrt{2}}\frac{2}{\sqrt{2}} = \frac{1}{2}. \\
\langle GHZ| P_- |GHZ\rangle &= \frac{1}{\sqrt{2}}\Big(\langle 000| + \langle 111| \Big)\Big(|-\rangle \langle -| \otimes I \otimes I\Big)\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle \Big) = \\
&= \frac{1}{2}\Big(\langle 000| + \langle 111| \Big)\Big(|-\rangle \overbrace{\langle -| |0\rangle}^{=\frac{1}{\sqrt{2}}} \otimes I |0\rangle \otimes I |0\rangle + |-\rangle \overbrace{\langle -| |1\rangle}^{=\frac{-1}{\sqrt{2}}} \otimes I |1\rangle \otimes I |1\rangle) \\
&= \frac{1}{2\sqrt{2}}\Big(\langle 000| + \langle 111| \Big)\Big(|-\rangle |00\rangle - |-\rangle |11\rangle \Big) = \\
&= \frac{1}{2\sqrt{2}}\Big(\overbrace{\langle 000| |-\rangle |00\rangle}^{=\frac{1}{\sqrt{2}}} + \overbrace{\langle 000| |-\rangle |11\rangle}^{=0} - \overbrace{\langle 111| |-\rangle |00\rangle}^{=0} - \overbrace{\langle 111| |-\rangle |11\rangle}^{=\frac{-1}{\sqrt{2}}} \Big) = \\
&= \frac{1}{2\sqrt{2}}\frac{2}{\sqrt{2}} = \frac{1}{2}.
\end{aligned}$$

And the post-measurement states are,

$$
\begin{aligned}
\frac{P_+ \left|GHZ\right\rangle}{\sqrt{\left\langle GHZ\right| P_+ \left|GHZ\right\rangle}} &= \frac{(\left|+\right\rangle \left\langle+\right| \otimes I \otimes I)\frac{1}{2}(\left|000\right\rangle + \left|111\right\rangle)}{\frac{1}{\sqrt{2}}} = \\
&= \frac{1}{\sqrt{2}}\Big(\left|+\right\rangle \left|00\right\rangle + \left|+\right\rangle \left|11\right\rangle \Big) = \\
&= \frac{1}{2}\Big(\left|000\right\rangle + \left|100\right\rangle + \left|011\right\rangle + \left|111\right\rangle \Big) \\
\frac{P_- \left|GHZ\right\rangle}{\sqrt{\left\langle GHZ\right| P_- \left|GHZ\right\rangle}} &= \frac{(\left|-\right\rangle \left\langle-\right| \otimes I \otimes I)\frac{1}{2}(\left|000\right\rangle + \left|111\right\rangle)}{\frac{1}{\sqrt{2}}} = \\
&= \frac{1}{\sqrt{2}}\Big(\left|-\right\rangle \left|00\right\rangle - \left|-\right\rangle \left|11\right\rangle \Big) = \\
&= \frac{1}{2}\Big(\left|000\right\rangle - \left|100\right\rangle - \left|011\right\rangle + \left|111\right\rangle \Big).
\end{aligned}
$$

- What are the possible post-measurement states and probabilities of them when measuring the first qubit in W state? Remember $\left|W\right\rangle = \frac{1}{\sqrt{3}}(\left|001\right\rangle + \left|010\right\rangle + \left|100\right\rangle)$.

We start with computing the probability of seeing 0 and 1.

$$
\begin{aligned}
\langle W| P_+ |W\rangle &= \frac{1}{\sqrt{3}}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|+\rangle \langle +| \otimes I \otimes I\Big)\frac{1}{\sqrt{3}}\Big(|001\rangle + |010\rangle + |100\rangle \Big) = \\
&= \frac{1}{3}\Big(\langle 001| + \langle 010| + \langle 100| \Big) \\
&\Big(|+\rangle \overbrace{\langle +| |0\rangle}^{=\frac{1}{\sqrt{2}}} \otimes I |0\rangle \otimes I |1\rangle + |+\rangle \overbrace{\langle +| |0\rangle}^{=\frac{1}{\sqrt{2}}} \otimes I |1\rangle \otimes I |0\rangle + |+\rangle \overbrace{\langle +| |1\rangle}^{=\frac{1}{\sqrt{2}}} \otimes I |0\rangle \otimes I |0\rangle \Big) \\
&= \frac{1}{3\sqrt{2}}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|+\rangle |01\rangle + |+\rangle |10\rangle + |+\rangle |00\rangle \Big) = \\
&= \frac{1}{3\sqrt{2}}\Big(\overbrace{\langle 001| |+\rangle |01\rangle}^{=\frac{1}{\sqrt{2}}} + \overbrace{\langle 001| |+\rangle |10\rangle}^{=0} + \overbrace{\langle 001| |+\rangle |00\rangle}^{0} + \\
&\overbrace{\langle 010| |+\rangle |01\rangle}^{=0} + \overbrace{\langle 010| |+\rangle |10\rangle}^{=\frac{1}{\sqrt{2}}} + \overbrace{\langle 010| |+\rangle |00\rangle}^{=0} + \\
&\overbrace{\langle 100| |+\rangle |01\rangle}^{=0} + \overbrace{\langle 100| |+\rangle |10\rangle}^{=0} + \overbrace{\langle 100| |+\rangle |00\rangle}^{=\frac{1}{\sqrt{2}}} \Big) = \\
&= \frac{1}{3\sqrt{2}}\frac{3}{\sqrt{2}} = \frac{1}{2}. \\
\langle W| P_- |W\rangle &= \frac{1}{\sqrt{3}}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|-\rangle \langle -| \otimes I \otimes I\Big)\frac{1}{\sqrt{3}}\Big(|001\rangle + |010\rangle + |100\rangle \Big) = \\
&= \frac{1}{3}\Big(\langle 001| + \langle 010| + \langle 100| \Big) \\
&\Big(|-\rangle \overbrace{\langle -| |0\rangle}^{=\frac{1}{\sqrt{2}}} \otimes I |0\rangle \otimes I |1\rangle + |-\rangle \overbrace{\langle -| |0\rangle}^{=\frac{1}{\sqrt{2}}} \otimes I |1\rangle \otimes I |0\rangle + |-\rangle \overbrace{\langle -| |1\rangle}^{=\frac{-1}{\sqrt{2}}} \otimes I |0\rangle \otimes I |0\rangle \Big) \\
&= \frac{1}{3\sqrt{2}}\Big(\langle 001| + \langle 010| + \langle 100| \Big)\Big(|-\rangle |01\rangle + |-\rangle |10\rangle - |-\rangle |00\rangle \Big) = \\
&= \frac{1}{3\sqrt{2}}\Big(\overbrace{\langle 001| |-\rangle |01\rangle}^{=\frac{1}{\sqrt{2}}} + \overbrace{\langle 001| |-\rangle |10\rangle}^{=0} + \overbrace{\langle 001| |-\rangle |00\rangle}^{0} + \\
&\overbrace{\langle 010| |-\rangle |01\rangle}^{=0} + \overbrace{\langle 010| |-\rangle |10\rangle}^{=\frac{1}{\sqrt{2}}} + \overbrace{\langle 010| |-\rangle |00\rangle}^{=0} - \\
&\overbrace{\langle 100| |-\rangle |01\rangle}^{=0} - \overbrace{\langle 100| |-\rangle |10\rangle}^{=0} - \overbrace{\langle 100| |-\rangle |00\rangle}^{=\frac{-1}{\sqrt{2}}} \Big) = \\
&= \frac{1}{3\sqrt{2}}\frac{3}{\sqrt{2}} = \frac{1}{2}.
\end{aligned}
$$

And the post-measurement states are ,

$$
\begin{aligned}
\frac{P_+ |W\rangle}{\sqrt{\langle W| P_+ |W\rangle}} &= \frac{(|+\rangle \langle +| \otimes I \otimes I)\frac{1}{\sqrt{3}\sqrt{2}}(|001\rangle + |010\rangle + |100\rangle)}{\frac{1}{\sqrt{2}}} = \\
&= \frac{1}{\sqrt{3}}\Big(|+\rangle |01\rangle + |+\rangle |10\rangle + |+\rangle |00\rangle \Big) = \\
&= \frac{1}{\sqrt{2}\sqrt{3}}\Big(|001\rangle + |101\rangle + |010\rangle + |110\rangle + |000\rangle + |100\rangle \Big) \\
\frac{P_- |W\rangle}{\sqrt{\langle W| P_- |W\rangle}} &= \frac{(|-\rangle \langle -| \otimes I \otimes I)\frac{1}{\sqrt{3}\sqrt{2}}(|001\rangle + |010\rangle + |100\rangle)}{\frac{1}{\sqrt{2}}} = \\
&= \frac{1}{\sqrt{3}}\Big(|-\rangle |01\rangle + |-\rangle |10\rangle - |-\rangle |00\rangle \Big) = \\
&= \frac{1}{\sqrt{2}\sqrt{3}}\Big(|001\rangle - |101\rangle + |010\rangle - |110\rangle - |000\rangle + |100\rangle \Big)
\end{aligned}
$$

F

Consider a general $|\psi\rangle = a\,|{++}\rangle + b\,|{+-}\rangle + c\,|{-+}\rangle + d\,|{--}\rangle$.

$$
\begin{aligned}
&|\psi\rangle\,|0\rangle = a\,|{++}\rangle + b\,|{+-}\rangle + c\,|{-+}\rangle + d{--}) \otimes |0\rangle \\
&\xrightarrow{(H\otimes H\otimes I)} (a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle) \otimes |0\rangle = a\,|000\rangle + b\,|010\rangle + c\,|100\rangle + d\,|110\rangle \\
&\xrightarrow{(CNOT_{1,3})} a\,|000\rangle + b\,|010\rangle + c\,|101\rangle + d\,|111\rangle \\
&\xrightarrow{(CNOT_{2,3})} a\,|000\rangle + b\,|011\rangle + c\,|101\rangle + d\,|110\rangle\,. \\
&\xrightarrow{(H\otimes H\otimes I)} a\,|{++}\rangle\,|0\rangle + b\,|{+-}\rangle\,|1\rangle + c\,|{-+}\rangle\,|1\rangle + |{--}\rangle\,|0\rangle = |\tilde{\psi}\rangle\,.
\end{aligned}
$$

Introducing the notation $|\tilde{\psi}\rangle$.

Since we want to measure only the third qubit in the $3 - \textit{qubit}$ state in the computational basis the corresponding projectors are: $P_0 = I \otimes I \otimes |0\rangle\langle 0|$ and $P_1 = I \otimes I \otimes |1\rangle\langle 1|$.

We start with computing the probability of seeing 0 and 1.

$$
\begin{aligned}
\langle\tilde{\psi}| P_0 |\tilde{\psi}\rangle &= \Big(a\langle ++|\langle 0| + b\langle +-|\langle 1| + c\langle -+|\langle 1| + d\langle --|\langle 0|\Big) \\
&\Big(I\otimes I\otimes |0\rangle\langle 0|\Big)\Big(a|++\rangle|0\rangle + b|+-\rangle|1\rangle + c|-+\rangle|1\rangle + d|--\rangle|0\rangle\Big) = \\
&= \Big(a\langle ++|\langle 0| + b\langle +-|\langle 1| + c\langle -+|\langle 1| + d\langle --|\langle 0|\Big) \\
&\Big(a\Big[I|+\rangle\otimes I|+\rangle\otimes|0\rangle\overbrace{\langle 0||0\rangle}^{=1}\Big] + b\Big[I|+\rangle\otimes I|-\rangle\otimes|0\rangle\overbrace{\langle 0||1\rangle}^{=0}\Big] + \\
&+ c\Big[I|-\rangle\otimes I|+\rangle\otimes|0\rangle\overbrace{\langle 0||1\rangle}^{=0}\Big] + d\Big[I|-\rangle\otimes I|-\rangle\otimes|0\rangle\overbrace{\langle 0||0\rangle}^{=1}\Big]\Big) = \\
&= \Big(a\langle ++|\langle 0| + b\langle +-|\langle 1| + c\langle -+|\langle 1| + d\langle --|\langle 0|\Big)\Big(a|++\rangle|0\rangle + d|--\rangle|0\rangle\Big) = \\
&= \ldots = \Big(a^2\langle ++|\langle 0||++\rangle|0\rangle + d^2\langle --|\langle 0||--\rangle|0\rangle\Big) = (a^2+d^2) \\
\langle\tilde{\psi}| P_1 |\tilde{\psi}\rangle &= \Big(a\langle ++|\langle 0| + b\langle +-|\langle 1| + c\langle -+|\langle 1| + d\langle --|\langle 0|\Big) \\
&\Big(I\otimes I\otimes |1\rangle\langle 1|\Big)\Big(a|++\rangle|0\rangle + b|+-\rangle|1\rangle + c|-+\rangle|1\rangle + d|--\rangle|0\rangle\Big) = \\
&= \Big(a\langle ++|\langle 0| + b\langle +-|\langle 1| + c\langle -+|\langle 1| + d\langle --|\langle 0|\Big) \\
&\Big(a\Big[I|+\rangle\otimes I|+\rangle\otimes|1\rangle\overbrace{\langle 1||0\rangle}^{=0}\Big] + b\Big[I|+\rangle\otimes I|-\rangle\otimes|-\rangle\overbrace{\langle +||1\rangle}^{=1}\Big] + \\
&+ c\Big[I|-\rangle\otimes I|+\rangle\otimes|1\rangle\overbrace{\langle 1||1\rangle}^{=1}\Big] + d\Big[I|-\rangle\otimes I|-\rangle\otimes|1\rangle\overbrace{\langle 1||0\rangle}^{=0}\Big]\Big) = \\
&= \Big(a\langle ++|\langle 0| + b\langle +-|\langle 1| + c\langle -+|\langle 1| + d\langle --|\langle 0|\Big)\Big(b|+-\rangle|1\rangle + c|-+\rangle|1\rangle\Big) = \\
&= \ldots = (b^2+c^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{P_0|\tilde{\psi}\rangle}{\sqrt{\langle\tilde{\psi}|P_0|\tilde{\psi}\rangle}} &= \frac{(I\otimes I\otimes|0\rangle\langle 0|)(a|++\rangle|0\rangle + b|+-\rangle|1\rangle + c|-+\rangle|1\rangle + d|--\rangle|0\rangle)}{\sqrt{a^2+d^2}} = \\
&= \frac{a|++\rangle|0\rangle + b|--\rangle|0\rangle}{\sqrt{a^2+d^2}} \\
\frac{P_1|\tilde{\psi}\rangle}{\sqrt{\langle\tilde{\psi}|P_1|\tilde{\psi}\rangle}} &= \frac{(I\otimes I\otimes|1\rangle\langle 1|)(a|++\rangle|0\rangle + b|+-\rangle|1\rangle + c|-+\rangle|1\rangle + d|--\rangle|0\rangle)}{\sqrt{b^2+c^2}} = \\
&= \frac{b|+-\rangle|1\rangle + c|-+\rangle|1\rangle}{\sqrt{b^2+c^2}}
\end{aligned}
$$